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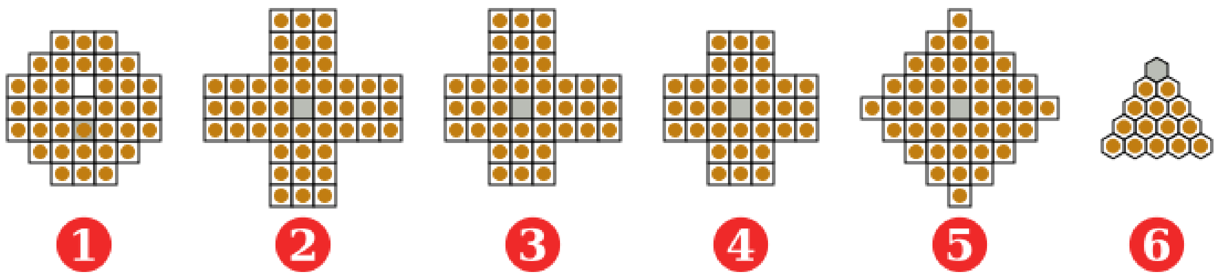
Investigations into 3-Color Peg Solitaire

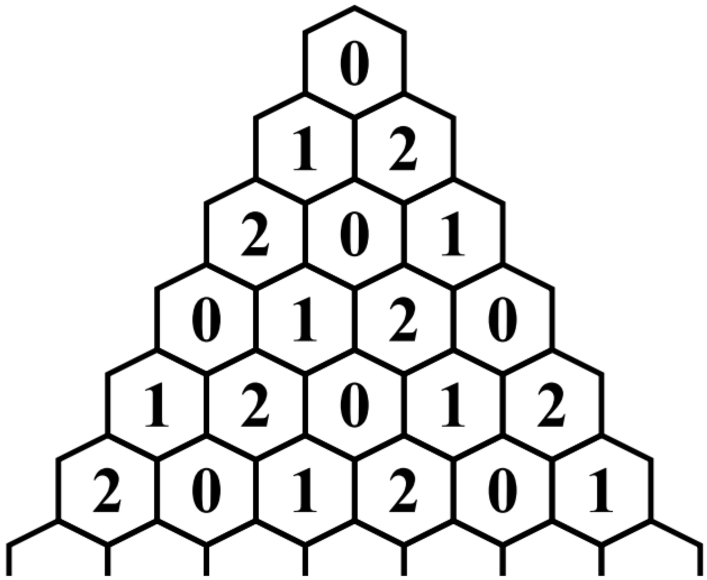
The game of peg solitaire has a rich and complex history. Many are familiar with the game and its seemingly simple rules: "jump" a peg and it disappears, and repeat until there is one peg remaining. However, a deeper analysis of peg solitaire reveals rich mathematical ideas behind both the strategy and solvability of the game. Boards can come in many different shapes; the most prominent of which have been listed below:



On the first 5 board layouts, moves are made by jumping pegs either horizontally or vertically. The 6th board, however, allows jumps in 6 different directions: one for each side of the hexagons. The solvability of these boards is well-established; however, less work has been done on different variations of the game. Our research focused specifically on peg solitaire in 3 colors on a triangle graph.

Analysis on traditional peg solitaire on triangle boards (board 6) can be found in the work of George Bell, who published a paper in 2008 titled "Solving Triangular Peg Solitaire." The standard rules for triangular peg solitaire involve a board containing all the holes filled with pegs, except for one empty starting hole. The game is then played by moves called "jumps," where a selected peg is jumped over an adjacent peg, and the peg that was jumped over is removed. When there are no more possible moves, the game is over. Traditionally, peg solitaire is understood to be won if the board state resolves to contain only one remaining peg at the conclusion of the game. Using this rule set, Bell discussed the theory of the game, and addressed the solvability (read: able to be reduced to a single peg) of various starting positions and board sizes. As we discuss the various arguments, it is important to understand the notation used. In the symbol T_n, the T represents a triangular board, while *n* represents the number of rows on the board. Coordinates on triangular boards are typically labeled in coordinate pairs of (*x, y*), where x is the 0-indexed vertical distance from the left side, and y is the 0-indexed distance from the top of the triangle (A/N: this is unintuitive, because the natural way to read a board is by starting at the top and counting down to the desired row. However, if done in this manner, the coordinate pairs will be read in the "wrong" order. Keep this in mind when reading coordinate pairs). A jump is typically represented by two coordinate pairs, one for the starting position of the moving peg and one for the ending position.

On a triangular board T_n where $n \geq 4$, one way to prove the solvability is a parity argument. Consider a board labeled as follows:



Bell, 2008.

In this diagram, the coordinates (x, y) are labeled (x + y mod 3). If a jump is executed on this board, it will necessarily involve 3 spaces with numerically different labels. Because the total number of pegs in two of the spaces will decrease in value (by the peg being moved and the peg being removed) and the total number of pegs in the other will increase in value, the overall parity of the sum of pegs in two selected spaces cannot change. If c_0 represents all pegs in spaces labeled 0, c_1 represents all pegs in spaces labeled 1, and c_2 represents all pegs in spaces labeled 2, then $(c_2 + c_1)$ will not change parity throughout the game (nor will any other combination of spaces). Because of this property, the vector created by $\{(c_0 + c_1), (c_0 + c_2), (c_2 + c_1)\}$ has 4 different possibilities, which partitions all board states into 4 equivalence classes: (0, 0, 0), (0, 1, 1), (1, 0, 1), and (1, 1, 0). Other combinations cannot occur, as it is impossible to have all three sums odd or exactly one sum odd. From this argument, Bell proves that for any T_n board with $n \geq$

4, the board is not solvable to one peg iff *n* = 1 (mod 3) and $x_s + y_s = 0$ (mod 3), where x_s and y_s represent the starting coordinates of the hole.

The T_5 board is what most people commonly think of when they hear of peg solitaire, due to its nationwide popularity at Cracker Barrel. Bell advances an interesting argument for the solvability of this board in particular, utilizing a pagoda function to track solvability. A *pagoda function* is defined in game theory as a number or value on a board that cannot increase as the game is played, providing an easy one-way method of tracking solvability from a known state. A synonymous term for a pagoda function is a *resource count*. Bell discusses an interesting pagoda function on the T_5 board named a board's "SAX" count, first discovered by Hentzel and Hentzel in 1986.

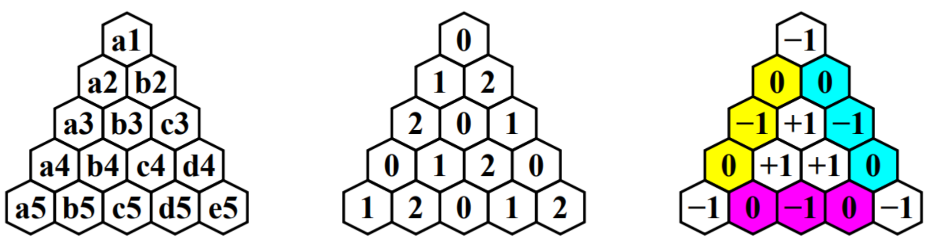


Figure 4: T_5 notation (a), position class (b), and the SAX count (c).

Bell, 2008.

The figure on the right contains a labeling of the board useful for understanding SAX count. Bell defines the following criteria:

- S is the number of colored edge regions with two or more pegs
- A is the number of pegs occupying holes labeled "+1"
- X is the number of pegs occupying holes labeled "-1"

With these restrictions, the overall SAX count of a board *cannot* increase during play. Thus, a board that starts with SAX count -2 is unsolvable. Furthermore, any board with starting SAX count -1 is also unsolvable. I encourage the reader to consider the board themselves to determine why these two statements are true (Dr. Grimley - is this allowed? I stole this technique from our Discrete Math textbook).

One final solution strategy for T_n boards is to consider *purges*, which are patterns of pegs that are always solvable. Bell highlights 3 patterns that are always solvable, and discusses how they can be grouped together to prove solvability. In the interest of brevity, these specific patterns will not be discussed here, for reasons soon to be addressed.

3-Color Boards

Our research interest was a variation on the traditional rules where the concept of "colors" is introduced to the graph. The normal game of peg solitaire can be conceptualized as having two colors: one for an empty hole and one for a peg. We added a third color to the game, representing a different color of peg. Under this new ruleset, if a peg is jumped over a peg of the same color, the peg that was jumped over switches to be the other color. If a peg is jumped over a peg of a differing color, the jumped peg is removed as normal:

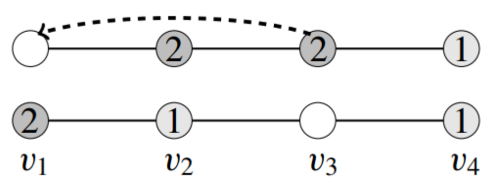


Figure 2. Example of moving a peg over a peg of the same color.

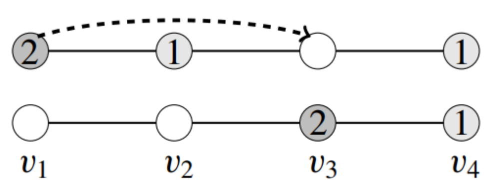


Figure 3. Example of moving a peg over a peg of a different color.

Davis et al., 2020.

Adding this ruleset drastically changes what we know about the solvability of different types of graphs. Davis et al. (2020) discusses some considerations when conceptualizing the board in this manner. Notably, they highlight the fact that moves can be represented in modular arithmetic: if the colors are labeled ‘0’ for an empty hole, and ‘1’ and ‘2’ for the two types of pegs, the result of v_2 for any jump $<v_1, v_2, v_3>$ is the value of $v_1 + v_2$ modulo 3. Similarly to the original game, the goal of this version is to have one peg remaining on the board once all moves have been conducted.

An important distinction to consider is the difference between a board being *solvable* and being *freely solvable*. A graph/board is considered solvable if there exists a solution path for at least one of its starting states. To be freely solvable, a board must contain a solution path for every starting state. Davis et al. (2020) show that path graphs (a board of just a straight line of pegs) are solvable, but not freely solvable. Consider the following board:

0 1 1 1 1 … 1 1

A move can be made where v_3 is hopped over v_2 to get the following result:

1 2 0 1 1 … 1 1

From there, v_1 can be hopped back over v_2 to obtain:

0 0 1 1 1 … 1 1

We now have the original board with length n - 1. This process can be repeated until the board is solved.

Using similar arguments, Davis et al. proves that cycle graphs are solvable for $n \geq 2$, bipartite graphs $K_{m,n}$ are freely solvable for all $m, n > 1$, star graphs $K_{1,n}$ are (n-1) solvable for $n > 2$. They also investigate Cartesian products of graphs and tree solvability. As our research focus was on the solvability of a triangular board in 3 colors, many of these proofs were not robustly investigated.

We investigated various strategies of proving the solvability of a triangle board of 3 colors. One initial problem that quickly became apparent was the issue that almost none of the strategies outlined by Bell (2008) of solving a triangle board were generalizable to 3 colors. For example, consider the aforementioned parity argument. Central to the proof is the fact that the total number of pegs in two of the three classes of holes decrease each jump, while the total number of pegs in the remaining class always increases. Unlike a traditional board, games in 3 colors do not necessarily follow this pattern. For jumps with two pegs of the same color, only one class of hole decreases in value, while one increases and the final class stays constant (due to the fact that the middle peg is not removed and only switches color). We investigated whether there were any other ways of conceptualizing parity, but this approach did not yield any results.

Similarly, we struggled to find an adequate resource count for triangle boards. The SAX count of a 3-color board does not fit the criteria, as it is possible for the SAX count to both increase and decrease during play. For example, consider the following boards:

```
    1
   1 2
  1 0 2
 2 2 1 2
2 2 1 1 2
SAX: -1
```

```
    1
   1 2
  1 2 2
 2 1 1 2
2 0 1 1 2
SAX: 0
```

The jump [(1, 4), (1, 3), (1, 2)] was performed, and the SAX count increased from -1 to 0. This discrepancy in how SAX count functions between 2-color boards and 3-color boards is again due to the fact that it is possible for pegs to not be removed for certain jumps. Once more, we investigated certain possibilities for alternative resource counts, but our efforts were fruitless.

We began looking into specific purge patterns for 3-color boards. Unsurprisingly, the patterns that Bell (2008) described did not generalize to our boards, and the specific purge patterns that we found to hold true for 3-color boards were not useful in the same manner. The conceptual basis for the utility of purges in traditional peg solitaire is that the ending state of the pegs is predictable: given a specific pattern of pegs, they can be solved to where the remaining peg will end in a desired location. One problem we encountered is that the ending location of the remaining peg was variable depending on the exact pattern of peg colors on the initial purge block. This made it difficult to fully string together purges to inductively prove the solvability of boards. However, this approach definitely holds potential, and we encourage future research to analyze purge removals in detail.

One possibility for why we found multiple solution strategies to be inapplicable to 3-color boards became apparent: were some boards of size T_n freely solvable? During our investigations, we noticed that we had not yet encountered an initial state for any T_5 board that was unsolvable. We then pivoted our focus to determine if all T_5 boards were freely solvable.

To answer this question, we wrote a program in Java designed to generate all starting positions of T_5 boards up to symmetry, and find a solution for them if possible. For T_5 boards, there are 4 symmetrically different starting positions for the initial hole. A proof for this will be included at the end of the paper. We utilized a simple recursive algorithm that checked all possible moves beginning at the top hole and worked downwards throughout the graph, outputting the first solution it found. For T_5 boards, all possible starting states contained at least one solution, demonstrating that our initial hypothesis was correct.

We then attempted to prove the solvability of other sizes of boards. Generating all solutions for T_4 boards displayed that not all starting states were solvable. Notably, all boards that started with a hole in the middle position were unsolvable:

```
   2
  2 2
 2 0 2
2 2 2 2
```

Regardless of the value of the pegs, a board of this configuration has no possible jumps, and thus is unsolvable. Select other T_4 boards were unsolvable as well, and the common theme for these boards was the middle hole necessarily appearing in their solution paths, and being unable to remove.

We then investigated T_6 boards to see what we could find. Unfortunately, due to the massive jump in possible configurations/solutions posed by the move from T_5 boards to T_6 boards, we were unable to feasibly generate all possible solutions. Our algorithm was allowed to run for approximately 6 hours, and output solutions for around ¼ of the possible T_6 boards. All boards generated were solvable, but more research is needed into proving if all starting states are.

Although there remains ground to cover before proving the solvability of certain triangular boards in 3 colors, we accomplished much during our research. We successfully investigated the applicability of previously known solutions strategies for traditional peg solitaire and discovered that they are generally inconclusive for boards in 3 colors. We also discovered which initial states of T_4 and T_5 boards were solvable. Now, the task is to determine exactly why. Time to get creative.

Proof of the number of starting positions up to symmetry on boards of size T_n:

Base case:

The board T_1 has one starting position:

```
  0
```

Although not much of a board, this case is important to consider.

The board T_2 also has one starting position, up to symmetry. As can be seen below, whichever position is selected to be '0' will always be able to be rotated so it matches this configuration:

```
  0
 1 1
```

The board T_3 has two starting positions, up to symmetry. As can be seen below, the starting hole has two options. It can either be on a corner:

```
  0
 1 1
1 1 1
```

Or on a position in the middle:

```
  1
 1 1
1 0 1
```

Inductive step:

Consider a board T_n where $n \geq 4$. Assume that for all boards T_k where $k < n$, the number of starting positions up to symmetry is given by the formula: $|T_k| = ceil(\frac{k}{2}) + |T_{k-3}|$, where $|T_{k-3}|$ represents the number of starting positions up to symmetry for the board T_{k-3}. The board T_n will necessarily contain a board of size T_{n-3} within itself, contained by a perimeter as shown below:

1
1 1
1 1 1
1 1 1 1
1 1 1 1 1
1 1 1 1 1 1
1 1 1 1 1 1 1

The perimeters of boards of size 1 (red), size 4(blue) and size 7 (pink).

Thus, the board of size T_n contains all the necessary different starting positions for symmetry of the board of size T_{n-3} alongside the additional starting positions created by the addition of a layer around it. Because the different sides of the triangle (and the individual halves of a specific side, rounded up) are symmetrically identical, the number of added starting positions up to symmetry is given by $ceil(\frac{n}{2})$. Because starting positions are additive, the number of starting positions of T_n is given by $|T_n| = ceil(\frac{n}{2}) + |T_{n-3}|$.

Because $|T_{n-3}| = 0$ for any boards of size $n < 1$, we know that T_1, T_2, and T_3 all follow this formula for symmetry. Thus, the proof is complete, and the number of starting positions up to symmetry for a board of size T_n is given by $|T_n| = ceil(\frac{n}{2}) + |T_{n-3}|$.

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